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Numerical Integration of Exact Time-Dependent Einstein Equations with Axial Symmetry. I*

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In the present Part I the complete system of exact equations describing a nonstationary state of an axially symmetric relativistic object consisting of a perfect fluid is reduced to the form suitable for the numerical integration by a computer. It is assumed that all the thermodynamic processes are adiabatic and no nuclear energy is being released, but no restriction is imposed on the analytically expressed equation of state. By the method it is possible to compute the initial data and the time evolution of the interior and exterior field generated either by a single rotating and contracting (or expanding) body, or by two neutron stars before and during their head-on collision. In Part II the method of the numerical integration will be described, its efficiency discussed, and a few examples of integration exhibited.

INTRODUCTION

A spherically symmetric star with an overcritical mass collapses into a black hole when its nuclear fuel is exhausted. However, since one can hardly imagine an astronomical object which exhibits no rotation at all with respect to the background cosmic field (in the case of a single object in the infinite empty space the body rotates with respect to the Minkowskian metric at infinity or, in the author's interpretation of Minkowskian metric [1], with respect to the infinite mass of Minkowskian universe distributed uniformly and isotropically with a zero density over the infinite cosmic space), a collapse of a spherically symmetric body is a process compatible with the Einstein field equations, and easy to handle from the mathematical standpoint, but which very probably never occurs in the actual Universe (like the geons [2]). The collapse of a star must be thus investigated taking into consideration its rotation whose angular velocity steadily increases during the contraction because of the conservation of the angular momentum and cannot be therefore considered as a small, linear, perturbation.

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A rotating and collapsing object is a source of gravitational radiation as well. Also two neutron stars before and during their collision emit gravitational waves. Under which conditions is this radiation strong enough to be observable by terrestrial detectors of gravitational waves?

Because of the extreme relativistic conditions a reliable answer to those two important questions of contemporary relativistic physics can result only from an investigation based on exact Einstein equations. The aim of the present paper is to show that with the help of a computer such an investigation is feasible, and in fact simple, in spite of the complicated form of the Ricci tensor.

The general relativity is distinguished by four important features: the equations of motion are the integrability conditions of the field equations; the field equations are invariant under general transformations of coordinates; in the harmonic coordinates the field equations for the empty space reduce in the weak field approximation to the homogeneous wave equations; there exists a certain mixed null tensor of the second rank whose four components depend upon the Cauchy data only. As a consequence, the complete system of equations describing the behaviour of a perfect fluid under assumption of adiabatic thermodynamic processes and no release of nuclear energy can be reduced to 6 independent Einstein field equations for 6 unknown components of the metric tensor and to one algebraic equation for the proper rest mass density (Sec. I); a system of comoving coordinates can be chosen in which the field equations can be numerically integrated in the most simple way (Sec. II, III); the Cauchy initial data problem can be properly formulated (Sec. V); and the four components of the null tensor which represent just the Lichnerowicz initial conditions (Sec. VII) serve as a sensitive and very important indicator for estimating the total amount of the roundoff and truncation errors in the process of the numerical integration of the 6 Einstein field equations (Sec. III).

The choice of the comoving coordinates is unavoidable for the present method of numerical integration of Einstein equations. This choice is connected, of course, with certain disadvantages; for instance, it is difficult to determine the geometrical dimensions of a rotating and contracting, or expanding, object. However, for comparison of the theory with observations this and similar questions are irrelevant; the explosion of a collapsing star and the distant radiation field, if strong enough, can be observed and measured, but not the radius of the object.

In order to demonstrate the clarity of the logical and mathematical structure of the general relativity theory appropriate substitutions are introduced in the main text which make the equations very simple. The explicit expressions for these substitutions which are given in the appendices are, of course, complicated enough, but not as much to make the numerical integration by a computer impossible.

In the preceding two papers [3, 4] the integration method was applied only in the region occupied by a rotating incoherent matter and the influence of the back-scattering of radiation upon the interior field was fully neglected. Since the results of those tests of the efficiency and limits of applicability of the integration method are very promising, the method is now extended to a perfect fluid with an arbitrary equation of state. The equations deduced below describe the interior and exterior field as well. The back-scattering of radiation and the interaction with the field of another body are also taken into consideration.

In the present paper a metric with signature $+2$ is assumed and a system of units used in which the velocity of light and the Newtonian constant of gravitation are equal to 1. A comma indicates a partial derivative, but where there is no danger of confusion the comma is omitted. The Riemannian derivative is denoted by a semicolon. Greek indices run from 1 to 4, Latin indices from 1 to 3.

I. Complete System of Equations for Non-Symmetric Fields

The complete system of 13 equations describing the behavior of a perfect fluid can be reduced in a particular system of comoving coordinates to 6 Einstein field equations and 1 equation of continuity [5, 6].[1] As prerequisites the reduction process will be here briefly summarized once again.

Let p indicate the pressure, ρ the proper rest mass density and ϵ the proper internal energy per unit mass. The four-velocity will be denoted by $u^\mu = dx^\mu/ds$, and the usual substitution

$$g = \det g_{\alpha\beta}$$

applied throughout.

The components of the energy-momentum tensor of the perfect fluid are defined by the expression

$$T_{\mu\nu} = e^\chi \rho u_\mu u_\nu + p g_{\mu\nu}\,, \tag{1.1}$$

in which

$$e^\chi = 1 + \epsilon + p/\rho \tag{1.2}$$

represents the enthalpy per unit mass.

Assuming that all the thermodynamic processes are adiabatic and no nuclear energy is being released, but supposing no geometric symmetry at all, the complete system of equations consists of one equation of state (of any given form in which the pressure vanishes when the mass density equals to zero)

$$p = p(\rho, \epsilon); \tag{1.3}$$

[1] The author thanks one of the referees for calling attention to the following two papers where Eqs. (1.6) and (1.7) also were integrated and equations deduced similar to those given in this Sec. I: A. H. Taub, "Fluids et Champ Gravitational en Relativité Générale," pp. 57 ff, Centre National de la Recherche Scientifique, Paris, 1969; B. F. Schultz, *Phys. Rev.* **D2** (1970), 2762.

of the conservation law of baryon number, which reduces in the case under consideration to the equation of continuity

$$(-g)^{-1/2}[\rho u^{\mu}(-g)^{1/2}]_{,\mu} = 0; \tag{1.4}$$

of the normalization equation for the four-velocity

$$g_{\mu\nu}u^{\mu}u^{\nu} = -1; \tag{1.5}$$

of $(1 + 3)$ equations of motion

$$T^{\nu}_{4;\nu} = 0, \tag{1.6}$$

$$T^{\nu}_{i;\nu} = 0, \tag{1.7}$$

and of 6 independent Einstein field equations

$$R_{ik} = 8\pi(T_{ik} - \tfrac{1}{2}Tg_{ik}). \tag{1.8}$$

The initial conditions for the metric tensor cannot be chosen quite arbitrarily, but they must satisfy four Lichnerowicz initial conditions [7]

$$\tfrac{1}{2}I_{\mu}{}^{4} \equiv R_{\mu}{}^{4} - \tfrac{1}{2}R\delta_{\mu}{}^{4} - 8\pi T_{\mu}{}^{4} = 0. \tag{1.9}$$

Since the system of 13 equations has to be satisfied by 17 functions, i.e. by 3 quantities of state, 4 components of the four-velocity, and 10 components of the metric tensor, four coordinate conditions must be now added. There is no doubt that the Ricci tensor takes a simpler form, say, in the synchronous reference frame of Landau and Lifshitz [8] or in the coordinate system recently used by Chandrasekhar and Friedman [9], but the author succeeded in reducing the preceding equations to a lower number only in a particular system of comoving coordinates. The applied method of numerical integration also requires the introduction of the comoving coordinates [4].

In the comoving coordinates, defined by 3 conditions

$$u^{i} = 0, \tag{1.10}$$

the only nonvanishing contravariant component u^4 of the four-velocity follows immediately from Eq. (1.5)

$$u^{4} = (-g_{44})^{-1/2} \tag{1.11}$$

and the equation of continuity (1.4) can be easily integrated giving

$$\rho = (g_{44}/g)^{1/2}\,\Psi(x^{j}) \tag{1.12}$$

where $\Psi(x^j)$ is determined by the initial distribution of the mass density. Substituting $\rho_{,4}$ from Eq. (1.4) into Eq. (1.6) and taking into account (1.10) and the fact that the equation of state (1.3) does not contain explicitly any coordinate x^μ, Eq. (1.6) may be now replaced by an equivalent, but simpler relation

$$d\epsilon/d\rho = p/\rho^2 \tag{1.13}$$

expressing the conservation of energy in the perfect fluid at a constant entropy (in the form familiar from elementary thermodynamics).

With the help of (1.13) and (1.12) the 3 equations of motion (1.7) reduce to the form

$$(g_{i4}/g_{44})(\partial/\partial x^4) \ln[g_{i4}e^\chi(-g_{44})^{-1/2}] - (\partial/\partial x^i) \ln[e^\chi(-g_{44})^{1/2}] = 0. \tag{1.14}$$

They can be integrated in a closed form if the condition

$$(\partial/\partial x^i) \ln[e^\chi(-g_{44})^{1/2}] = 0 \tag{1.15a}$$

is chosen as the fourth coordinate condition. The integration of (1.15a) yields

$$g_{44} = -[A_4(x^4)\, e^{-\chi}]^2. \tag{1.15b}$$

$A_4(x^4)$ being an arbitrary function. As a consequence of (1.15a), Eq. (1.14) gives after integration

$$g_{i4} = -A_i A_4 e^{-2\chi} \tag{1.16}$$

where $A_i(x^j)$ are three functions of spatial coordinates determined by the initial conditions.

For a given equation of state (1.3) the pressure p and the enthalpy e^χ may be now considered as known functions of the mass density which is determined by the algebraic equation (1.12). Since the components $g_{\mu 4}$ are given by (1.15b) and (1.16), the remaining unknown functions are the 6 components g_{ik} of the metric tensor determined by the 6 field equations (1.8).

The covariant components of the four-velocity take now the values

$$u_\mu = u^4 g_{\mu 4} = -A_\mu e^{-\chi}. \tag{1.17}$$

With their help the components $g_{\mu 4}$ may be also expressed in the form

$$g_{\mu 4} = -u_\mu u_4\,. \tag{1.18}$$

Since the enthalpy e^χ is a scalar, A_μ is a four-vector. Similar to u^4, it has only one nonvanishing contravariant component

$$A^4 = -u^4 e^\chi = -e^{2\chi}/A_4\,. \tag{1.19}$$

The four-vector of vorticity is defined by the formula [10]

$$\Omega^{\mu} = \tfrac{1}{2}(-g)^{-1/2}\, \epsilon^{\mu\alpha\beta\gamma} u_{\alpha} u_{\beta,\gamma}\,, \qquad \epsilon^{\mu\alpha\beta\gamma} = \pm 1, 0. \tag{1.20a}$$

It reduces with the help of Eq. (1.17) to

$$\Omega^{\mu} = \tfrac{1}{2}(-g)^{-1/2}\, \epsilon^{\mu\alpha\beta\gamma} A_{\alpha} A_{\beta,\gamma} e^{-2x}. \tag{1.20b}$$

This vector is curl of A_{μ} ; therefore it is perpendicular to A^{μ} [7]:

$$\Omega^{\mu} A_{\mu} = 0 \tag{1.21}$$

The motion of matter is irrotational if the tensor of vorticity

$$\tfrac{1}{2}(A_{i,k} - A_{k,i})$$

vanishes. In this case it is always possible by a coordinate transformation to reduce all the A_i to zero.

II. Choice of the Coordinate System

Without any loss of generality the choice of the coordinate system, limited already by Eqs. (1.10) and (1.15b), will be now restricted by further conditions in order to make the explicit expressions for the Ricci tensor as simple as possible and to make easier also the solution of the difficult problem of Lichnerowicz initial conditions. This particular coordinate system was applied in the preceding two papers [3, 4] without proof that its applications did not imply a loss of generality. In this section the missing proof is given.

The $(3 + 1)$ coordinate conditions (1.10) and (1.15b) restrict the allowed transformations of coordinates to the following ones

$$\bar{x}^{i} = \bar{x}^{i}(x^1, x^2, x^3), \tag{2.1}$$

$$\bar{x}^{4} = \bar{x}^{4}(x^4). \tag{2.2}$$

By the transformation (2.2) it is always possible to reduce $| A_4 |$ to 1, but then the allowed transformations are restricted to those defined by (2.1) and by

$$\bar{x}^{4} = \pm x^4 + \text{const}. \tag{2.3}$$

The vector A^{μ} has the direction of the time axis x^4 and is always perpendicular to the vorticity vector Ω^{μ} [cf. Eqs. (1.19) and (1.21)]. By the transformation (2.1) it is thus possible to reduce an arbitrary system of three spatial coordinates to an orthogonal system whose x^1 direction coincides at any point with the direction of the vector Ω^μ. In the new coordinate system it holds at any moment [11] that

$$\bar{\Omega}^1 \neq 0, \qquad \bar{\Omega}^2 = \bar{\Omega}^3 = \bar{\Omega}^4 = 0, \tag{2.4}$$

and that [cf. Eq. (1.20b)]

$$\bar{A}_1 = 0, \qquad \bar{A}_2 = \bar{A}_2(\bar{x}^2, \bar{x}^3), \qquad \bar{A}_3 = \bar{A}_3(\bar{x}^2, \bar{x}^3), \qquad |\bar{A}_4| = 1, \tag{2.5}$$

but

$$\bar{g}_{12} = \bar{g}_{13} = \bar{g}_{23} = 0 \tag{2.6}$$

only at $\bar{x}^4 = 0$.

By the subsequent transformation

$$\bar{x}^1 = \bar{\bar{x}}^1, \qquad \bar{x}^2 = \bar{x}^2(\bar{\bar{x}}^2, \bar{\bar{x}}^3), \qquad \bar{x}^3 = \bar{x}^3(\bar{\bar{x}}^2, \bar{\bar{x}}^3), \tag{2.7}$$

the component $\bar{\bar{A}}_2$ can be reduced at any moment to zero

$$\bar{\bar{A}}_2 \equiv \bar{A}_2(\partial \bar{x}^2/\partial \bar{\bar{x}}^2) + \bar{A}_3(\partial \bar{x}^3/\partial \bar{\bar{x}}^2) = 0,$$

the axis $\bar{\bar{x}}_1$ remains everywhere orthogonal to $\bar{\bar{x}}^2$ and to $\bar{\bar{x}}^3$, but since $\bar{g}_{22}$ and $\bar{g}_{33}$ depend, in general, on $\bar{x}^1$, the component

$$\bar{\bar{g}}_{23} = \bar{g}_{22}(\partial \bar{x}^2/\partial \bar{\bar{x}}^2)(\partial \bar{x}^2/\partial \bar{\bar{x}}^3) + \bar{g}_{33}(\partial \bar{x}^3/\partial \bar{\bar{x}}^2)(\partial \bar{x}^3/\partial \bar{\bar{x}}^3)$$

can vanish only for one chosen value of $\bar{x}^1 = \bar{\bar{x}}^1$, say $\bar{\bar{x}}^1 = 0$. The axis $\bar{\bar{x}}^2$ remains thus orthogonal to $\bar{\bar{x}}^3$ only at $\bar{\bar{x}}^1 = 0$. It now holds (the double bars are omitted) that

$$A_1 = A_2 = 0, \qquad A_3 = A_3(x^2, x^3), \qquad |A_4| = 1, \tag{2.8}$$

at any moment, and at the initial moment only

$$g_{12} = g_{13} = 0 \qquad \text{everywhere}, \tag{2.9a}$$

$$g_{23} = 0 \qquad \text{at} \qquad x^1 = 0. \tag{2.9b}$$

The transformations of coordinates are now restricted to

$$\bar{x}^1 = \bar{x}^1(x^1), \qquad \bar{x}^2 = \bar{x}^2(x^2), \qquad \bar{x}^3 = \bar{x}^3(x^3), \tag{2.10}$$

and to Eq. (2.3). The reduction of A_μ to (2.8) simplifies the Ricci tensor, the reduction of the initial values of $g_{\mu\nu}$ to (2.9a,b) simplifies the solution of the Lichnerowicz initial conditions.

The field of the vorticity filaments, represented by the lines $x^2 = \text{const.}$, $x^3 = \text{const.}$, determines thus the direction of the x^1 axis. Its position is given by the condition that at it the x^2-component of the pressure gradient vanishes. If the field exhibits an axial symmetry, its axis is chosen, of course, as the x^1 axis.

In the numerical integration the unknown functions g_{ik} are computed at equally spaced grid points. The exterior field at large distances from the body generating the field varies in space as well as in time more smoothly than the interior field. The geometrical distances between the grid points thus may be chosen far greater at the periphery of the integration domain than inside the body. This can be achieved by the coordinate transformations (2.10) which reduce g_{11} and g_{22} at the initial moment to

$$g_{11} = e^{2\alpha^*(x^1)} \quad \text{at} \quad x^2 = 0, \tag{2.11a}$$

$$g_{22} = e^{2\tilde{\beta}(x^2)} \quad \text{at} \quad x^1 = 0, \quad x^3 = \text{const.}, \tag{2.11b}$$

where $\alpha^*(x^1)$ and $\tilde{\beta}(x^2)$ are two properly chosen functions ($\alpha^* = \tilde{\beta} = 0$ at the periphery of the integration domain, $\alpha^* < 0, \tilde{\beta} < 0$ inside the body—cf. Sec. VII). If the field is axially symmetric, the component g_{22} takes the chosen values at the whole hyperplane $x^1 = 0$.

The conditions (2.11a,b) and the requirement that the metric has to be Euclidean in the infinitesimal neighborhood of the x^1 axis restrict the allowed transformations of coordinates

$$\begin{aligned} \bar{x}^1 &= \pm x^1 + \text{const.}, & \bar{x}^2 &= x^2, \\ \bar{x}^3 &= \pm x^3 + \text{const.}, & \bar{x}^4 &= \pm x^4 + \text{const.}, \end{aligned} \tag{2.12}$$

to the translation of the origin of the coordinates x^1, x^3, and x^4, and to the reflections in these three axes. The position of the origin of the coordinate system follows usually from the geometrical symmetry in the integration domain. The initial distribution of the mass density and vorticity together with Eqs. (2.8), (2.9a,b) and (2.11a,b) determine uniquely and in the most natural way the comoving coordinate system. It is no more necessary to carry out the important, but sometimes difficult (in the case of numerical integration almost impossible), investigation whether a given metric tensor corresponds to a new physical situation or has been generated from a known one by a mere transformation of coordinates.

III. Axially Symmetric Field Equations

In this and in the following sections the field is supposed to be axially symmetric. This assumption simplifies, of course, the equations, but the main reason of this restriction lies in the fact that the grid of points where the field is numerically computed is two-dimensional, while it must be three-dimensional when the field exhibits no geometrical symmetry at all. As a consequence, the computer time would become too long in the latter case.

The computer program for the numerical differentiation of the components of the metric tensor would be far more simple if a system of spherical coordinates were chosen, but in this case none of the components A_i could be reduced to zero, and what is decisive, the use of the spherical coordinates would be to advantage only if the field generated by a single body were investigated. Therefore the cylindrical coordinates (z, r, ϕ, t) are preferred, with the metric in the form

$$ds^2 = e^{2\alpha}\, dz^2 + e^{2\beta}\, dr^2 + (e^{2\eta} - A^2 e^{-2\chi})\, d\phi^2 - e^{-2\chi}\, dt^2 + 2U\, dz\, dr + 2V\, dz\, d\phi + 2W\, dr\, d\phi - 2Ae^{-2\chi}\, d\phi\, dt. \tag{3.1}$$

The unknown functions α, β, η, U, V, W, χ depend on z, r, t, and, in agreement with Eq. (2.8),

$$A = A(r). \tag{3.2}$$

It is very important to take $g_{33} = (e^{2\eta} - A^2 e^{-2\chi})$, because then Eq. (1.12) determines ρ as an algebraic function of $e^{2\alpha}$, $e^{2\beta}$, $e^{2\eta}$, U, V, W, but not of χ, which depends on the equation of state and may thus be a complicated function of ρ.[2]

The determinant of the metric tensor and its contravariant components are to be computed first in the usual way.[3] For the calculation of the components of the Ricci tensor forms have been used [12] which are based on the formula

$$R_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(g_{\mu\alpha,\nu\beta} + g_{\nu\beta,\mu\alpha} - g_{\mu\nu,\alpha\beta} - g_{\alpha\beta,\mu\nu}) + g^{\alpha\beta} g^{\gamma\delta}[(\gamma, \mu\alpha)(\delta, \nu\beta) - (\gamma, \mu\nu)(\delta, \alpha\beta)]$$

involving only algebraic operations and Christoffel symbols of the first kind

$$(\epsilon, \mu\nu) = \tfrac{1}{2}(g_{\epsilon\mu,\nu} + g_{\epsilon\nu,\mu} - g_{\mu\nu,\epsilon}).$$

With the help of the forms the calculation of $R_{\mu\nu}$ is easy; possibly it does not require more time than translating the corresponding results obtained by a computer from Fortran language into the usual mathematical symbols. The subsequent reduction of these results to a compact form (which must be carried out in order to keep the computer time as short as possible) takes more time and hardly can be done by the computer.

The metric (3.1) is Euclidean in the infinitesimal neighborhood around the z axis if

$$\eta(z, r, t) = \sigma(z, r, t) + \ln r \tag{3.3a}$$

and

$$\sigma = \beta \quad \text{at} \quad r = 0. \tag{3.3b}$$

[2] Cf. the last equation on p. 285 in [6].

[3] Cf. Eqs. [3.3]–[3.6] in [3].

The vorticity vector Ω^1 is finite at the z axis if

$$A = r^2 a(r) = r^2 a^* e^{2\nu(r)}, \qquad a^* = \text{const.}, \qquad \nu(0) = 0. \tag{3.4}$$

Its square, given by the relation

$$|\,\Omega\,|^2 = g_{\alpha\beta}\Omega^\alpha\Omega^\beta, \tag{3.5}$$

is identical with the square of the angular velocity measured ($c = 1$) by a local observer.

In a preceding paper [3] it has been shown that the field is regular at the z axis at least at the initial moment and at its infinitesimal past and future only if[4]

$$U = ru, \qquad V = r^2 v, \qquad \text{and} \qquad W = r^3 w, \tag{3.6}$$

and if the functions α, β, σ, u, v, w, χ, ρ, ν, are even functions in r. If the mass exhibits also a reflection symmetry with respect to the hyperplane $z = 0$, the functions α, β, σ, w, χ, ρ, ν, are even functions in z, while u and v are odd functions in z.

After the functions η, U, V, and W have been replaced by σ, u, v, and w by Eqs. (3.3a,b) and (3.6), the determinant of the metric tensor and its contravariant components as well as the Ricci tensor take a slightly different form. Appendix A contains the formulas for g and $g^{\mu\nu}$. The functions $k^{\mu\nu}$ defined in Appendix A are not tensors; their superscripts just indicate in which relation they stand to the contravariant components of the metric tensor.

The 6 independent Einstein field equations (1.8) may be now expressed as follows:

$$\alpha_{44} = \{4\pi\rho(e^\chi - 2p/\rho) + P_{11}\}/e^{2\chi}k^{44} \tag{3.7a}$$

$$\beta_{44} = \{4\pi\rho(e^\chi - 2p/\rho) + P_{22}\}/e^{2\chi}k^{44} \tag{3.7b}$$

$$\sigma_{44} = \{4\pi\rho[(e^\chi - 2p/\rho) + a^2r^2e^{-2\sigma-2\chi}(e^\chi + 2p/\rho)] + P_{33}\}/e^{2\chi}k^{44} - e^{-2\sigma-2\chi}r^2a^2\chi_{44} \tag{3.7c}$$

$$u_{44} = \{8\pi\rho(e^\chi - 2p/\rho)u + P_{12}\}/e^{2\chi}k^{44} \tag{3.7d}$$

$$v_{44} = \{8\pi\rho(e^\chi - 2p/\rho)v + P_{13}\}/e^{2\chi}k^{44} \tag{3.7e}$$

$$w_{44} = \{8\pi\rho(e^\chi - 2p/\rho)w + P_{23}\}/e^{2\chi}k^{44} \tag{3.7f}$$

Henceforth subscripts at the functions α, β, σ, u, v, w, χ, ρ, ν indicate the partial derivatives; the comma is omitted here. The functions $P_{\mu\nu}$ are defined in Appendix B together with the substitutions introduced to simplify the formulas for $P_{\mu\nu}$. The $P_{\mu\nu}$ are not tensors; their subscripts just indicate in which relation they stand to the covariant components of Ricci tensor. In computing $P_{\mu\nu}$ the relation

$$k^{44} + e^{-2\sigma-2\chi}k^{33}a^2r^2 = 1 \tag{3.8}$$

[4] The functions v and w are defined here in a slightly different way from those in [3].

following from the formulas of Appendix A was repeatedly used. As a consequence, one finds terms in $P_{\mu\nu}$ that are formed by a product of two first derivatives of the metric tensor, but contain one coefficient $k^{\mu\nu}$ (or none). The metric (3.1) as well as the functions $P_{\mu\nu}$ are invariant with respect to the following simultaneous permutations:

$$z \rightleftarrows r, \qquad 1 \rightleftarrows 2, \qquad \alpha \rightleftarrows \beta, \qquad r^2 v \rightleftarrows r^3 w.$$

This invariance is a powerful means for eliminating possible errors of the computation.[5] However, because the function A does not depend on the z coordinate, the terms containing the derivatives $A_{,2}$ and $A_{,22}$ have no counterpart and must be therefore checked especially carefully.

Since the components g_{ik} ($i \neq k$) vanish at the z axis as given by Eqs. (3.6), the field is here regular. In spite of it the functions $P_{\mu\nu}$ contain terms that have an indeterminate form 0/0, or $\infty - \infty$. Therefore two sets of the functions $P_{\mu\nu}$ must be used: one set for the space with $r > 0$, and the other for the z axis and denoted by an asterisk, $P^*_{\mu\nu}$, in which the indeterminancy is analytically evaluated. The third derivatives occurring in $P^*_{\mu\nu}$ are the result of the limiting process $r \to 0$, for instance,

$$\lim_{r\to 0} \sigma_{42}/r = \sigma^*_{422} \,.$$

(The asterisk indicates here, and henceforth, the value of the function at $r = 0$).

The geometrical properties of space are described by the positive definite metric of Landau and Lifshitz [8]

$$d\bar{s}^2 = \gamma_{ik}\, dx^i\, dx^k \tag{3.9a}$$

with

$$\gamma_{ik} = g_{ik} - g_{i4} g_{k4}/g_{44}\,, \qquad \gamma^{ik} = g^{ik}. \tag{3.9b}$$

In the metric (3.1) all the components of γ_{ik} are identical with g_{ik} with the exception of

$$\gamma_{33} = g_{33} + A^2 e^{-2\chi} = e^{2\eta} = r^2 e^{2\sigma}. \tag{3.10}$$

The mass density ρ is determined in the $g_{\mu\nu}$ metric by the equation of continuity (1.12). In the γ_{ik} metric Eq. (1.12) reduces to

$$\rho = \Psi/(\det \gamma_{ik})^{1/2}. \tag{3.11a}$$

It is thus the expansion, or contraction, of space described by the three-dimensional metric (3.9a,b) which directly determines the proper rest mass density. The explicit form of (3.11a) is

$$\rho = \Psi K^{1/2} e^{-\alpha-\beta-\sigma}, \tag{3.11b}$$

[5] The invariance of the functions $P_{\mu\nu}$ with respect to the simultaneous permutations had been checked before the functions η, U, V, W were replaced by σ, u, v, w.

where K is given by the formula of Appendix A. The function $\overline{\Psi}$ depending on the spatial coordinates only is computed by Eq. (3.11b) from the initial data of the metric and mass density.

The functions $P_{\mu\nu}$ and $P^*_{\mu\nu}$ and the field equation (3.7c) contain derivatives of the function χ. The differentiation of Eq. (1.2) with the relation (1.13) taken into account yields

$$\chi_\mu = (\rho_\mu/\rho)E, \tag{3.12a}$$

$$\chi_{\mu\nu} = (\rho_{\mu\nu}/\rho)E + (\rho_\mu/\rho)(\rho_\nu/\rho)F, \tag{3.12b}$$

$$\chi^*_{\mu 22} = (\rho^*_{\mu 22}/\rho^*)E + (\rho_\mu{}^*/\rho^*)(\rho^*_{22}/\rho^*)F, \tag{3.12c}$$

where the functions E and F depend upon the equation of state

$$E = e^{-\chi}(dp/d\rho), \tag{3.13a}$$

$$F = e^{-\chi}[\rho(d^2p/d\rho^2) - (dp/d\rho)(1 + E)]. \tag{3.13b}$$

The third derivatives $\chi^*_{\mu 22}$ (with $\mu = 1, 4$) occur only in $P^*_{\mu\nu}$. The formula for $\chi^*_{\mu 22}$ has the form (3.12c) because $\rho_2{}^* = 0$ (for ρ is always an even function in r).

The derivative (ρ_4/ρ) results from the differentiation of Eq. (3.11a):

$$\rho_4/\rho = -k^{11}\alpha_4 - k^{22}\beta_4 - k^{33}\sigma_4 + e^{-2\alpha-2\beta}k^{12}r^2u_4 + e^{-2\alpha-2\sigma}k^{13}r^2v_4 + e^{-2\beta-2\sigma}k^{23}r^4w_4\,. \tag{3.14}$$

Hence

$$\chi_{4i} = E(\partial/\partial x^i)(\rho_4/\rho) + (E + F)(\rho_4/\rho)(\rho_i/\rho), \tag{3.15a}$$

$$\chi^*_{422} = E[(\partial^2/\partial r^2)(\rho_4/\rho)]_{r=0} + (E + F)(\rho_4{}^*/\rho^*)(\rho^*_{22}/\rho^*). \tag{3.15b}$$

One differentiation more of Eq. (3.14) with respect to the timelike coordinate gives

$$\rho_{44}/\rho = (\rho_4/\rho)^2 - 2r^2P_\rho - k^{11}\alpha_{44} - k^{22}\beta_{44} - k^{33}\sigma_{44} + e^{-2\alpha-2\beta}k^{12}r^2u_{44} + e^{-2\alpha-2\sigma}k^{13}r^2v_{44} + e^{-2\beta-2\sigma}k^{23}r^4w_{44}\,, \tag{3.16}$$

with P_ρ defined in Appendix B. The field equation (1.8) with $\mu = \nu = 4$ may be written in the form

$$-k^{11}\alpha_{44} - k^{22}\beta_{44} - k^{33}\sigma_{44} + e^{-2\alpha-2\beta}k^{12}r^2u_{44} + e^{-2\alpha-2\sigma}k^{13}r^2v_{44} + e^{-2\beta-2\sigma}k^{23}r^4w_{44} - e^{-2\sigma-2\chi}k^{33}a^2r^2\chi_{44} - P_{44} = 4\pi\rho e^{-2\chi}(e^\chi + 2p/\rho) \tag{3.17}$$

The function P_{44} is also defined in Appendix B. Combining these two equations together with (3.12b) and (3.8) results in the formula

$$\chi_{44} = [1 + E(k^{44} - 1)]^{-1} \times \{(E + F)(\rho_4/\rho)^2 + E[4\pi\rho e^{-2x}(e^x + 2p/\rho) + P_{44} - 2r^2 P_\rho]\} \quad (3.18)$$

Once the equation of state (1.3) is given and Eq. (1.13) is integrated, the enthalpy e^x, the pressure p, as well as the functions E and F, defined by Eqs. (3.13a,b), may be considered as known functions of the mass density ρ. The derivatives of χ are reduced by Eqs. (3.12a,b,c), (3.14), (3.15a,b) and (3.18) to the derivatives of the mass density. The derivatives of any unknown function occurring in P_{ik} with respect to spatial coordinates may be expressed, using Lagrange formulas for numerical differentiation, by the function itself at the given point and at its neighborhood [4]. If the cross section, $\phi =$ const., of the integration domain is divided into a two-dimensional grid of n equally spaced points where the unknown functions α, β, σ, u, v, w, and ρ are to be calculated, the set of 6 partial differential equations (3.7a–f) may be now considered as a set of $6n$ simultaneous ordinary differential equations and integrated using, for instance, the fourth-order Runge-Kutta method. The algebraic equation (3.11b) determines at each point the seventh unknown function ρ. However, in each computation of the right-hand sides of (3.7a–f) all the g^{ik} and all the derivatives must be evaluated anew and the integration must be carried out for all $6n$ functions simultaneously.

The finite number n of the grid points is a source of truncation errors in the computation of spatial derivatives. Another source of truncation errors is the integration method for ordinary differential equations. The general relativity yields, however, a sensitive indicator for estimating the total amount of all the errors of the numerical calculation: If the four components of $I_\mu{}^4$ given by (1.9) vanish at the initial moment, then they vanish also at any moment $t \lesseqgtr 0$ [7]. Due to the numerical errors the $I_\mu{}^4$ will differ from zero at $t \gtrless 0$, and this difference may be used as a criterion of the total amount of errors, for it is highly improbable that the errors could partially cancel each other in the $I_\mu{}^4$ in such a way that the order of magnitude of the $I_\mu{}^4$ would be smaller than the order of magnitude of all the errors.

The components $I_\mu{}^4$ may be expressed by the functions $P_{\mu\nu}$:

$$I_1{}^4 = e^{2x}k^{44}P_{14} - e^{-2\sigma}r^2 a(2k^{13}P_{11} + e^{-2\beta}k^{23}r^2 P_{12} - k^{33}P_{13}), \quad (3.19a)$$

$$I_2{}^4 = e^{2x}k^{44}P_{24} - e^{-2\sigma}r^3 a(e^{-2\alpha}k^{13}P_{12} + 2k^{22}P_{22} - k^{33}P_{23}), \quad (3.19b)$$

$$I_3{}^4 = e^{2x}k^{44}P_{34} - r^2[e^{-2\sigma}r^2 a(e^{-2\alpha}k^{13}P_{13} + e^{-2\beta}k^{23}r^2 P_{23}) - 2k^{33}P_{33} - 16\pi p e^x], \quad (3.19c)$$

$$I_4{}^4 = e^{2x}k^{44}P_{44} + k^{11}P_{11} + k^{22}P_{22} + k^{33}P_{33} - e^{-2\alpha-2\beta}k^{12}r^2 P_{12} - e^{-2\alpha-2\sigma}k^{13}r^2 P_{13} - e^{-2\beta-2\sigma}k^{23}r^4 P_{23} + 16\pi\rho(e^x - p/\rho). \quad (3.19d)$$

At $r = 0$ the formulas for $I_\mu{}^4$ reduce to

$$I_1^{4*} = 4e^{2\chi^*}\{\beta_{41}^* + e^{-2\beta^*}[u^*\alpha_4{}^* - \tfrac{1}{2}u_4{}^* + e^{-2\beta^*-2\chi^*}a^*v^*] + \beta_1{}^*(\beta_4{}^* - \alpha_4{}^*) + \beta_4{}^*\chi_1{}^*\}, \tag{3.20a}$$

$$I_2^{4*} = I_3^{4*} = 0, \tag{3.20b,c}$$

$$I_4^{4*} = e^{2\chi^*}P_{44}^* + P_{11}^* + P_{22}^* + P_{33}^* + 16\pi\rho^*(e^{\chi^*} - p^*/\rho^*). \tag{3.20d}$$

Besides the existence of the four components $I_\mu{}^4 = 0$, the general relativity exhibits another important difference against Newtonian dynamics. In the latter the acceleration of a mass element dm is given by the global distribution of the mass and, eventually, by the fictitious forces. Because of the hyperbolic character of the Einstein field equations the second timelike derivatives in Eqs. (3.7a–f), which may be considered as a relativistic counterpart of the classical acceleration, are determined by the local value of the mass density and of the curvature of space-time expressed by the functions P_{ik} (and, in Eq. (3.7c), also by χ_{44}). In the P_{ik} the whole past gravitational history is hidden, but the present values of P_{ik}, compatible with Eq. (19), may be such that the future time evolution is completely different, even if the field is weak, from that according to Newtonian dynamics.

In a system of noncomoving coordinates the functions P_{ik} have a far simpler form [9]. However, it is only in the comoving coordinates that the complete system [illegible] equation (3.11b) and the partial derivatives with respect to the spatial coordinates may be computed by the Lagrange differentiation formula in which the interval between two subsequent equally spaced points does not change during the whole process of the numerical integration [4].

IV. Junction Conditions and Exterior Field

Let Σ be a smooth hypersurface separating the interior field from the exterior field of the empty space. The hypersurface itself is a part of both subdomains. In the comoving cylindrical coordinates of the interior metric the hypersurface is given by the equation

$$S(z, r) = 0. \tag{4.1}$$

According to Lichnerowicz [7], the junction conditions require the continuity of the metric tensor and of its normal derivatives across the hypersurface Σ if the metric is expressed in the admissible coordinates. However, Synge [13] has shown that even the nonadmissible coordinates may be used on both sides of the hypersurface Σ, providing that they are obtained from the admissible ones by a C^1 transformation[6] and that the following four junction conditions are satisfied

$$(G_\mu{}^\nu S_{,\nu})_{\text{interior}} = (G_\mu{}^\nu S_{,\nu})_{\text{exterior}} \qquad \text{at } \Sigma, \tag{4.2}$$

$G_\mu{}^\nu$ being the Einstein tensor. The components $g_{\mu\nu}$ are still continuous across Σ (since their transformation law involves only the first derivatives $\partial x^\alpha/\partial \bar{x}^\beta$), but their first normal derivatives $g_{\mu\nu,\epsilon}$ may be now discontinuous.

The Einstein field equations, written now in the form

$$G_\mu{}^\nu \equiv R_\mu{}^\nu - \tfrac{1}{2}R\delta_\mu{}^\nu = 8\pi T_\mu{}^\nu, \tag{4.3}$$

show that the right-hand side of (4.2) vanishes, because the exterior space is assumed to be empty. Since $S_{,\nu} = 0$ if $\nu = 3, 4$, and since in the interior domain the comoving coordinates are used for which Eq. (1.10) holds, the energy momentum tensor, given by Eq. (1.1), vanishes if

$$p = 0 \qquad \text{at } \Sigma. \tag{4.4}$$

Consequently, the Einstein tensor of the interior metric vanishes at Σ too if Eq. (4.4) is satisfied. In the comoving coordinates four junction conditions (4.2) reduce thus to one condition (4.4).

In the time evolution the mass density varies according to Eq. (3.11b). If the condition (4.4) has to be satisfied at any moment, the pressure in the equation of state (1.3) must vanish when the mass density vanishes. This restriction imposed upon the equation of state is certainly quite irrelevant for the behavior of a collapsing object. The condition (4.4) reduces now to

$$\overline{\Psi}(z, r) = 0 \qquad \text{at } \Sigma. \tag{4.5}$$

In the exterior domain the harmonic coordinates, defined by the four coordinate conditions [14]

$$(-\bar{g})^{-1/2}(\partial/\partial\bar{x}^\nu)[\bar{g}^{\mu\nu}(-\bar{g})^{1/2}] = 0 \tag{4.6}$$

must be used if the Einstein field equations[7]

$$R^{\mu\nu} = 0 \tag{4.7}$$

have to reduce in the distant weak field to the homogeneous wave equations

$$\Box \bar{h}^{\mu\nu} = 0 \tag{4.8}$$

in which

$$\bar{h}^{\mu\nu} = \bar{g}^{\mu\nu} - \bar{\eta}^{\mu\nu} \tag{4.9}$$

[6] The meaning of this restriction is simple: It guarantees the length of an arbitrary vector to be the same in all the coordinate systems that are related to each other by the C^1 transformations.

[7] It is advantageous to use the contravariant components in Eq. (4.7) if the harmonic coordinates are applied, because the expression for $R^{\mu\nu}$ is then simpler. Cf. [14], pp. 192–194.

denote the deviations of $\bar{g}^{\mu\nu}$ from the Minkowskian values $\bar{\eta}^{\mu\nu}$. The asymptotic form of the coordinate conditions (4.6) is

$$(\bar{h}^{\mu\nu} - \tfrac{1}{2}\bar{h}\bar{\eta}^{\mu\nu})_{,\nu} = 0, \qquad \bar{h} = \bar{h}^{\alpha\beta}\bar{\eta}_{\alpha\beta}\,. \tag{4.10}$$

The d'Alembertian operator $\Box$ in (4.8) may be expressed in any chosen coordinate system (for instance, in spherical coordinates), but Eq. (4.10) holds only in the harmonic coordinates $\bar{x}^{\mu}$, and further, the $\bar{g}^{\mu\nu}$, $\bar{h}^{\mu\nu}$, $\bar{\eta}^{\mu\nu}$ must be the components of tensors in the harmonic coordinates.[8]

Two successive transformations from the interior comoving coordinates x^{μ} to the admissible ones and from them to the harmonic coordinates $\bar{x}^{\mu}$ may be replaced by a single transformation given by the equation [14]

$$\Box\bar{x}^{\mu} \equiv (-g)^{1/2}(\partial/\partial x^{\alpha})[(-g)^{1/2}\, g^{\alpha\beta}(\partial\bar{x}^{\mu}/\partial x^{\beta})] = 0. \tag{4.11}$$

The six boundary values of $\bar{g}^{ik}$,

$$\bar{g}^{ik} = g^{\alpha\beta}(\partial\bar{x}^{i}/\partial x^{\alpha})(\partial\bar{x}^{k}/\partial x^{\beta}), \tag{4.12}$$

are the source of the gravitational radiation generated by the object inside the two-dimensional surface $S(z, r) = 0$. The remaining four components $\bar{g}^{\mu 4}$ are given by the coordinate conditions (4.6).

When one or more bodies forming an insular system are surrounded by the empty space-time, Eq. (4.11) and the boundary conditions (4.12) must be satisfied at the hypersurface Σ_i of each body if the back-scattering of radiation [15] and the gravitational interaction are to be taken into account. It is obvious that such a difficult boundary value problem can hardly be either solved analytically or programed for a computer.

After a thorough examination of the alternatives the author does not see any practicable other approach than the following one: Let Λ be a spherical hypersurface where the exterior field is so weak that the exact equations (4.7) and (4.6) may be replaced with a sufficient accuracy by the approximate ones (4.8) and (4.10). The body, or the insular system of bodies moving along the z axis to preserve the supposed axial symmetry, are assumed not to be surrounded by the empty space-time, but the whole domain inside Λ is now considered as the interior field where the mass density and the metric tensor are C^3 continuous and expressed in the comoving coordinates of Sec. III. The mass density takes very low values outside the bodies and equals zero at Λ. This is in fact a better description of the actual physical situation in the interstellar space than the assumption that the interstellar

[8] This is an analogue to the wave equation of the electromagnetic theory $\Box E_i = 0$. The d'Alembertian operator may be here expressed in arbitrary coordinates, but the E_i must be the Cartesian components of the vector **E**. The harmonic coordinates are thus in this sense a generalization of the Cartesian coordinates of Euclidean geometry.

space is empty. In this way the many body problem is reduced to one body problem and the boundary between the interior field and the exterior field of the empty space is shifted to the region where the interior metric differs from the Minkowskian by very small quantities $h^{\mu\nu}$ so that the equation (4.11) with the background metric reduces here to the wave equation of Euclidean geometry with the particular solution

$$\bar{x}^1 \equiv \bar{z} = z, \qquad \bar{x}^2 \equiv \bar{x} = r\cos\phi,$$
$$\bar{x}^3 \equiv \bar{y} = r\sin\phi, \qquad \bar{x}^4 \equiv \bar{t} = t. \tag{4.13}$$

The harmonic coordinates outside Λ are thus identical with the Cartesian coordinates.

From ten components $\bar{h}^{\mu\nu}$ only $\bar{h}^{44}$ is of importance; the other nine components are negligibly small. Since the interior coordinates are transformed to the exterior ones by Eqs. (4.13), the h^{44} and h_{44} of the interior metric are related at Λ to $\bar{h}^{44}$ and $\bar{h}_{44}$ of the exterior metric by

$$\bar{h}^{44} = h^{44}, \qquad \bar{h}_{44} = h_{44}. \tag{4.14}$$

Since the indices are lowered and raised at Λ and in the exterior domain by the background metrics, it holds also that

$$h^{44} = -h_{44}, \qquad \bar{h}^{44} = -\bar{h}_{44}. \tag{4.15}$$

V. Cauchy Initial Data

Because of the transient character of the generation of gravitational waves the initial values of $\bar{g}^{\mu\nu}$ at the entire hypersurface $\bar{t}$ = const. of the empty space never can be known, for in them the whole past gravitational history of the objects generating the waves is contained. The $\bar{g}^{\mu\nu}$ must, of course, satisfy the Lichnerowicz initial conditions (1.9), but this is a minor restriction. An improper choice of the initial values $\bar{g}^{\mu\nu}$ might imply the presence of gravitational waves generated not by the bodies, but somewhere at infinity.

However, the Einstein equations are quasilinear hyperbolic differential equations of second order for the integration of which the Cauchy data at the initial hypersurface $t = \bar{t} = 0$ must be given. Fortunately, in the weak field zone the field equations reduce to the homogeneous wave equations for which the classical Huygens principle [16, 17] holds. The principle asserts that sharp signals are transmitted in three-dimensional space as sharp signals, that is, that the solution of the wave equations describing the propagation of signals emitted at $t = 0$ depends upon the data at the boundary of the conoid of dependence, not upon the data inside. The principle implies that the signals are transmitted only in the direction of the propagation of waves, but not in the opposite direction, for the propagation towards the source of radiation would cause reverberation and make the transmission of sharp signals impossible (this occurs in the space of even number of dimensions).

Consequently, if the space outside Λ is assumed to be empty and with no gravitational waves incoming from infinity, then the initial data inside Λ, satisfying the initial conditions (1.9), represent the Cauchy initial data which determine the whole past and future gravitational history of all the bodies inside Λ. The gravitational waves propagating towards infinity are to be computed either with the help of the Huyghens principle from the field at the wave front Ξ or with the help of the Fourier transform from the field at Λ. There exists no back-scattering at the wave front Ξ lying in the immediate vicinity of Λ and, consequently, no back-scattering at the hypersurface Λ (Sec. VI).

Cauchy initial data alone cannot exclude the presence of a short pulse wave inside Λ which was generated at infinity and focussed onto this domain. However, if such a pulse wave were present, then the solution of the field equations (3.7a–f) for the interior domain would be determined not only by Cauchy initial data inside Λ, but also by the boundary values at Λ during the time interval when the incoming pulse wave was crossing the hypersurface Λ. If the integral of Eqs. (3.7a–f) is determined by Cauchy initial data inside Λ only, then this fact implies the absence of whatever incoming radiation.

VI. Weak Field Zone

In order to determine the relation between the boundary values at the spherical hypersurface Λ, given by the equation

$$r - r_\Lambda = 0, \qquad r_\Lambda = \text{const.}, \tag{6.1}$$

and the distant exterior weak field described in the spherical coordinates (r, θ, ϕ, t), let us assume that the interior problem has been solved in the whole interval $-\infty \leqslant t \leqslant \infty$ so that the boundary values $\bar{h}_\Lambda^{\mu\nu}(\theta, t)$ are known. The $\bar{h}_\Lambda^{\mu\nu}$ may be expanded into a series of spherical harmonics:

$$\bar{h}_\Lambda^{\mu\nu}(\theta, t) = \sum_{n=0}^{\infty} A_n^{\mu\nu}(t)\, P_n(\cos\theta). \tag{6.2}$$

The Fourier transform of $A_n^{\mu\nu}(t)$ may be denoted by $a_n^{\mu\nu}(f)$:

$$a_n^{\mu\nu}(f) = \int_{-\infty}^{+\infty} A_n^{\mu\nu}(t)\, e^{2\pi i f t}\, dt. \tag{6.3}$$

The unknown components $\bar{h}^{\mu\nu}(r, \theta, t)$ are now given by the superposition of the well-known particular solutions of the linear wave equation (4.8) in spherical coordinates:

$$\bar{h}^{\mu\nu} = \sum_{n=0}^{\infty} P_n(\cos\theta) \Big\{ \int_0^{\infty} [H_n^{(1)}(2\pi f r)/H_n^{(1)}(2\pi f r_A)]\, a_n^{\mu\nu}(f)\, e^{-2\pi i f t}\, df$$

$$+ \int_{-\infty}^{0} [H_n^{(2)}(2\pi \mid f \mid r)/H_n^{(2)}(2\pi \mid f \mid r_A)]\, a_n^{\mu\nu}(f)\, e^{-2\pi i f t}\, df \Big\}, \qquad (6.4)$$

where

$$H_n^{(1)}(z) = (e^{iz}/z)\, e^{-i(\pi/2)(1+n)} \sum_{k=0}^{n} (-2iz)^{-k}[(n+k)!/k!(n-k)!], \qquad (6.5a)$$

$$H_n^{(2)}(z) = (e^{-iz}/z)\, e^{i(\pi/2)(1+n)} \sum_{k=0}^{n} (2iz)^{-k}[(n+k)!/k!(n-k)!], \qquad (6.5b)$$

are the spherical Hankel functions. Equation (6.4) must, of course, contain only outgoing waves. Therefore the spherical Hankel functions $H_n^{(1)}(z)$ are used in Eq. (6.4) if $f \geqslant 0$, and $H_n^{(2)}(\mid z \mid)$ if $\leqslant 0$.

Since $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ are complex functions of a real argument, the functions $\bar{h}^{\mu\nu}$ given by Eq. (6.4) remain always finite for $z > 0$, with $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ approaching

$$\lim_{z\to\infty} H_n^{(1)}(z) = (e^{iz}/z)\, e^{-i(\pi/2)(1+n)}, \qquad (6.6a)$$

$$\lim_{z\to\infty} H_n^{(2)}(z) = (e^{-iz}/z)\, e^{i(\pi/2)(1+n)}. \qquad (6.6b)$$

The radiation field tends thus to zero with increasing distance from the radiation hypersurface A for whatever boundary values, but the field retains its axial symmetry even at infinity.

If

$$2\pi \mid f \mid r_A \equiv 2\pi r_A/\lambda \gg 1, \qquad (6.7)$$

λ being the wavelength of the gravitational radiation, then it holds approximately that

$$H_n^{(1)}(2\pi f r)/(H_n^{(1)}(2\pi f r_A) \cong (r_A/r)\, e^{2\pi i f (r-r_A)} \qquad (6.8a)$$

$$H_n^{(2)}(2\pi \mid f \mid r)/H_n^{(2)}(2\pi \mid f \mid r_A) \cong (r_A/r)\, e^{2\pi i f (r-r_A)} \qquad \text{if } f < 0. \qquad (6.8b)$$

The substitution of these relations into Eq. (6.4) and the application of Fourier integral theorem yields

$$\bar{h}^{\mu\nu}(r, \theta, t) \cong (r_A/r) \sum_{n=0}^{\infty} A_n^{\mu\nu}(t - [r - r_A])\, P_n(\cos\theta). \qquad (6.9)$$

Comparing Eq. (6.9) with Eq. (6.2) one finds

$$\bar{h}^{\mu\nu}(r, \theta, t) \cong (r_\Lambda/r)\, \bar{h}_\Lambda^{\mu\nu}(\theta, t - [r - r_\Lambda]). \tag{6.10}$$

The author is aware that as $|f| \to 0$ the approximate relations (6.8a,b) must not be applied, because they do not satisfy the condition (6.7). However, the sensitivity of a detector of gravitational waves is frequency-dependent. The coefficients $a_n^{\mu\nu}(f)$ in Eqs. (6.4) must be therefore multiplied by the sensitivity function of the detector in order to obtain, in Eqs. (6.9) and (6.10), the observable part of the gravitational radiation. As long as the radius r_Λ of the hypersurface Λ is much greater than the longest wave length to which the detector is sensitive, the approximate relations (6.10) may be used. The weak radiation field (or, at least, its observable part) thus propagates, in the terminology of Courant and Hilbert [16], as a "relatively undistorted" progressive wave.

In order to determine the distant radiation field it is thus sufficient to expand the only relevant component $h_{44} = -(e^{-2x} - 1) = -\bar{h}_\Lambda^{44}$ at Λ (cf. Eqs. (4.14) and (4.15) into a series of Legendre polynomials (6.2) with the coefficients

$$A_n^{44}(t) = (\tfrac{1}{2} + n) \int_0^\pi \bar{h}_\Lambda^{44}(\theta, t)\, P_n(\cos\theta) \sin\theta \, d\theta. \tag{6.11}$$

The distant radiation field is then given by Eq. (6.9).

Concluding, let us remark that the wave equation (4.8) reduces to the Laplace equation and the coefficients $A_n^{\mu\nu}$ in (6.2) are constant if the exterior field is generated by a stationary metric at Λ. The stationary exterior field is in this case given by the superposition of the well-known particular solutions of Laplace's equation in spherical coordinates:

$$\bar{h}^{\mu\nu}(r, \theta) = \sum_{n=0}^{\infty} (r_\Lambda/r)^{1+n} A_n^{\mu\nu} P_n(\cos\theta). \tag{6.12}$$

If $r \gg r_\Lambda$, it holds approximately that

$$\bar{h}^{\mu\nu}(r, \theta) \cong (r_\Lambda/r)\, A_0^{\mu\nu}. \tag{6.13}$$

The distant stationary field becomes thus always spherically symmetric with $A_0^{\mu\nu}$ is given by the formula

$$A_0^{\mu\nu} = \tfrac{1}{2} \int_0^\pi \bar{h}^{\mu\nu}(\theta) \sin\theta\, d\theta. \tag{6.14}$$

VII. Lichnerowicz Initial Conditions

The solution of Lichnerowicz initial conditions (1.9) is far more difficult than the time evolution problem of Sec. III. In Sec. II it has been shown that without loss of generality the functions u and v may be put equal to zero at the initial moment everywhere and to w at $z = 0$ (cf. Eqs. (2.9a,b) and (3.6)). In this section the equations (1.9) will be not written down in full generality, because they are too complicated, but only for the case when at $t = 0$

$$u = v = w = 0 \qquad \text{everywhere.} \tag{7.1}$$

The meaning of this restriction is simple: At $t = 0$ all the vorticity filaments are curves lying in the Euclidean planes $\phi = \text{const}$. If the matter does not rotate, Eq. (7.1) represents no restriction of generality.

Equation (7.1) is now substituted into Eqs. (3.19a–d) and into the formulas of Appendix B. The initial conditions (1.9) take thus the form:

$$\begin{aligned} I_1{}^4 e^{-2x}/2k^{44} \equiv\ & \beta_{41} + \sigma_{41} - \alpha_4(\beta_1 + \sigma_1) + \beta_4(\beta_1 + \chi_1) \\ & + \sigma_4(\sigma_1 + \chi_1) - (b_{00}/k^{44})\{\sigma_{41} + \chi_{41} \\ & - (\sigma_1 + \chi_1)(\alpha_4 - \beta_4 - \sigma_4 + \chi_4)\} - \tfrac{1}{2}e^{-2\beta}\{ru_{42} + u_4[1 + \alpha_0 - \beta_0 \\ & + \sigma_0 + \chi_0 + (2b_{00}/k^{44})(\sigma_0 + \chi_0 - 2a_0)]\} = 0; \end{aligned} \tag{7.2a}$$

$$\begin{aligned} I_2{}^4 e^{-2x}/2k^{44} \equiv\ & \alpha_{42} + \sigma_{42} + \alpha_4(\alpha_2 + \chi_2) - \beta_4(\alpha_2 + \sigma_2) \\ & + \sigma_4(\sigma_2 + \chi_2) + (\sigma_4 - \beta_4)/r + (b_{00}/k^{44})\{\sigma_{42} + \chi_{42} \\ & + (\sigma_2 + \chi_2)(\alpha_4 - \beta_4 + \sigma_4 - \chi_4) - (2a_0/r)(\alpha_4 - \beta_4 - 2\chi_4) \\ & + (\alpha_4 - \beta_4 + \sigma_4 - \chi_4)/r\} \\ & - \tfrac{1}{2}e^{-2\alpha}r\{u_{41} - u_4[\alpha_1 - \beta_1 - (\sigma_1 + \chi_1)(1 + 2b_{00}/k^{44})]\} = 0; \end{aligned} \tag{7.2b}$$

$$\begin{aligned} I_3{}^4 e^{2\alpha-2x}/k^{44}r^2 \equiv\ & v_4[\alpha_1 - \beta_1 - (\sigma_1 + \chi_1)(1 + 2b_{00}/k^{44})] \\ & - v_{41} + e^{2\alpha-2\beta}\{w_4[r(\beta_2 - \alpha_2 - \sigma_2 - \chi_2) - 4 \\ & - (2b_{00}/k^{44})(r\sigma_2 + r\chi_2 + 1 - 2a_0)] - rw_{42}\} \\ & + (2ae^{-2x}/k^{44})\{\sigma_{11} + \chi_{11} - (\sigma_1 + \chi_1)(\alpha_1 - \beta_1 - \sigma_1 + \chi_1) \\ & + e^{2\alpha-2\beta}[\sigma_{22} + \chi_{22} - a_{00}(1 + b_{00}) + (\alpha_2 - \beta_2 + 2\sigma_2)/r \\ & + (\sigma_2 + \chi_2)(\alpha_2 - \beta_2 + \sigma_2 - \chi_2) - (4b_{00}a_0{}^2/r^2) \\ & + a_0(1 + b_{00})(\beta_2 - \alpha_2 + \sigma_2 + 3\chi_2)/r]\} + 16\pi\rho ae^{2\alpha-x}/k^{44} = 0; \end{aligned} \tag{7.2c}$$

$$
\begin{aligned}
&(2e^{2\alpha+2\sigma}/r^2)[(I_3{}^4e^{-2\chi})(ae^{-2\sigma-2\chi}/k^{44}) + I_4{}^4e^{-2\chi}] \\
&\quad \equiv v_4{}^2 - (4ae^{-2\chi}/k^{44})(\sigma_1 + \chi_1)\, v_4 + e^{2\sigma-2\beta}k^{44}u_4{}^2 \\
&\quad - (4e^{2\alpha+2\sigma}/r^2)\{k^{44}\alpha_4\beta_4 + (\alpha_4 + \beta_4)(\sigma_4 + b_{00}\chi_4)\} \\
&\quad + e^{2\alpha-2\beta}w_4\{w_4r^2 - (4ae^{-2\chi}/k^{44})[1 - a_0 + r(\sigma_2 + \chi_2) - a_0b_{00}]\} \\
&\quad + (4e^{2\sigma-2\chi}/r^2)\{\beta_{11} + \beta_1(\beta_1 - \alpha_1) + (1/k^{44})[\sigma_{11} + \sigma_1(\beta_1 + \sigma_1 - \alpha_1)] \\
&\quad + (b_{00}/k^{44})[\chi_{11} + \chi_1(\beta_1 - \alpha_1 - \chi_1)]\} \\
&\quad + (4e^{2\alpha+2\sigma-2\beta-2\chi}/r^2)\big(\alpha_{22} + \alpha_2(\alpha_2 - \beta_2) \\
&\quad + (1/k^{44})\{\sigma_{22} - 2a_{00}b_{00} + \sigma_2(\alpha_2 - \beta_2 + \sigma_2) + (\alpha_2 - \beta_2 + 2\sigma_2)/r \\
&\quad + 2a_0\gamma[r(\beta_2 - \alpha_2 + \sigma_2) - \tfrac{1}{2}a_0(3 + b_{00})]\} \\
&\quad + (b_{00}/k^{44})[\chi_{22} + \chi_2(\alpha_2 - \beta_2 - \chi_2 + 6a_0/r)]\big) \\
&\quad + (32\pi\rho e^{2\alpha+2\sigma-2\chi}/r^2)[(e^{\chi}/k^{44}) - (p/\rho)] = 0; \qquad (7.2\text{d})
\end{aligned}
$$

where

$$\gamma = a^2e^{-2\sigma-2\chi}, \qquad (7.3)$$

$$b_{00} = 1 - k^{44}. \qquad (7.4)$$

The condition $I_4{}^4 = 0$ has been replaced in Eq. (7.2d) by a linear combination of $I_3{}^4$ and $I_4{}^4$ to make the equation simpler.

It is advantageous to choose α_4, β_4, v_4, w_4 as unknown functions of (z, r), because only the first derivatives of these functions with respect to z or r occur in [redacted]

$$\alpha_4 = \beta_4 = \sigma_4 = \rho_4 = 0; \qquad (7.5)$$

i.e., when the functions α, β, σ, and the mass density ρ have just reached their extremum values. Equation (3.14) shows that in this case (since $w_4 \neq 0$)

$$w = 0 \qquad \text{everywhere.} \qquad (7.6\text{a})$$

The unique solution of Eqs. (7.2a,b) is now

$$u_4 = 0 \qquad \text{everywhere.} \qquad (7.6\text{b})$$

The remaining unknown functions v_4 and w_4 are determined by the quasilinear differential equation of first order (7.2c) and by the quadratic algebraic equation (7.2d).

In the preceding paper [18] dealing with the stationary equilibrium of relativistic rotating objects it has been shown that in this equilibrium the conditions (7.1), (7.5), (7.6a,b), and $v_4 = 0$, are satisfied. Supposing that the resulting equations in

[18] have been solved either exactly or approximately (for instance, under assumption of a slowly rotating star), we may now consider the equilibrium to be slightly disturbed, say, due to a change of the equation of state. In this way we obtain a set of the initial data satisfying the conditions (7.1), (7.5), and (7.6a,b), and corresponding to an initial situation of a collapsing rotating star. For these data the functions v_4 and w_4 are now to be computed in order to satisfy the initial conditions (7.2c,d).

Equation (7.2d) may be rewritten in the form

$$v_4^2 - 2Lv_4 + e^{2\alpha-2\beta}r^2 w_4(w_4 - 2M) - N = 0. \tag{7.7}$$

The functions L, M, N, as well as the functions L^*, L_1, $L_1{}^*$, M^*, $\frac{1}{2}N_1$, P^*, $P_1{}^*$, Q^*, S^* introduced later, are defined in Appendix C. The subscript "1" denotes the partial derivative with respect to z; the asterisk indicates the values of the function at $r = 0$ (for instance, $L^* = L(z, 0)$); the functions P^* and $P_1{}^*$ stand in no relation to the functions $P_{\mu\nu}$, $P^*_{\mu\nu}$, P_ρ defined in Appendix B.

Equation (7.7) can be satisfied at $r = 0$ only if the sum of terms involving r^{-2} in N vanishes. To fulfill this requirement we choose

$$\begin{aligned}\alpha^*_{22}(z) \equiv G^*(z) = \tfrac{1}{2}\beta^*_{22} - (3/2)(\sigma^*_{22} - \gamma^*)\\ - e^{2\beta^*-2B^*}\{\beta^*_{11} + \beta_1{}^*[(3/2)\,\beta_1{}^* - B_1{}^*]\} - 4\pi\rho^* e^{2\beta^*}(e^{\chi^*} - p^*/\rho^*).\end{aligned} \tag{7.8}$$

The functions B^* and γ^* are defined by

$$B^* = \alpha^*(z), \tag{7.9}$$

and (cf. Eqs. (7.3) and (3.3b)) by

$$\gamma^* = a^{*2}e^{-2\beta^*-2\chi^*}. \tag{7.10}$$

We now assume that the mass density exhibits a reflection symmetry with respect to the hyperplane $z = 0$. The function v_4 is thus an odd function in z. Equation (7.7) shows that at $z = r = 0$ it must hold that

$$\lim_{r\to 0} N(0, r) \equiv P^*(0) - (8/3)\, e^{2B^*(0)-2\chi^*(0)}\alpha^*_{2222}(0) = 0.$$

Hence

$$\alpha^*_{2222}(0) \equiv C = (3/8)\, e^{2\chi^*(0)-2B^*(0)}P^*(0) = \text{const.} \tag{7.11}$$

Equation (7.7) determines first the values of w_4 at $z = r = 0$

$$w_4(0, 0) = M^*(0) \pm \{[M^*(0)]^2 - 4e^{2\beta^*(0)-2B^*(0)-2\chi^*(0)}S^*(0)\}^{1/2} \tag{7.12}$$

where $S^*(0)$ is defined by the formula

$$S^*(0) = -\tfrac{1}{4}e^{2x^*(0)} \lim_{r\to 0} N(0, r)/r^2, \tag{7.13}$$

and, explicitly, in Appendix C.

The value of $w_4(0, 0)$ is also given by the differential equation (7.2c). The requirement that they must be equal to each other yields the integrability condition

$$\alpha^*_{112222}(0) \equiv D = 3e^{-2B^*(0)}\{a^{*2}e^{-2x^*(0)}[\beta^*_{11}(0) + \chi^*_{11}(0)]^2 - Q^*(0)\}$$
$$- 12e^{2B^*(0)}\{[w_4(0, 0) - M^*(0)]\, e^{-2\beta^*(0)+x^*(0)} - 4\pi\rho^*(0)\, a^*\}^2 = \text{const.} \tag{7.14}$$

$Q^*(0)$ is defined by the relation following from Eq. (7.7) in which the fact has been taken into account that v_4, as an odd function in z, vanishes at $z = 0$:

$$\begin{aligned} v_{41}(0, 0) &= L_1^*(0) \pm \lim_{\substack{z\to 0\\ r\to 0}} (\partial/\partial z)(L^2 + N)^{1/2} \\ &= L_1^*(0) \pm \{\lim_{\substack{z\to 0\\ r\to 0}} [(L^2 + N)/z^2]\}^{1/2} \\ &= L_1^*(0) \pm \{[L_1^*(0)]^2 - 4e^{-2x^*(0)}[Q^*(0) + (1/3)\, e^{2B^*(0)}\alpha^*_{112222}(0)]\}^{1/2}. \end{aligned} \tag{7.15}$$

Since $v_4(z, r)$ and $w_4(z, r)$ have been chosen as unknown functions, Eqs. (7.2c,d), together with the assumption that the mass density is distributed with reflection symmetry with respect to $z = 0$, restrict the free choice of the initial values of the functions α, β, σ. In the preceding calculation the restriction is put upon the function α. It is given by the formula

$$\alpha = B^*(z) + (1/2)\, r^2G^*(z) + (1/24)\, r^4C + (1/48)\, z^2r^4D + z^4r^4q(z, r) + r^6s(z, r). \tag{7.16}$$

The functions $B^*(z)$, $q(z, r)$, $s(z, r)$, as well as $\beta(z, r)\, \sigma(z, r)$, $a(r)$, may be freely chosen with the restriction that they must be even functions in z and in r and that the discriminant in Eq. (7.12), as well as in Eqs. (7.18), (7.20), and (7.23) deduced below, does not become negative in the whole integration domain. We choose B^*, q, s, β, σ equal to zero at Λ and negative inside the bodies and at their close surroundings so that the point grid used in the numerical integration becomes denser here (the functions must be of course, at least C^3 continuous). $G^*(z)$ is given by Eq. (7.8).

The function $\bar{\Psi}(z, r)$ may be now computed from Eq. (3.11b):

$$\bar{\Psi} = \rho e^{\alpha+\beta+\sigma}. \tag{7.17}$$

The mass density must, of course, vanish at the spherical hypersurface Λ.

The value of w_4 at $z = r = 0$ is already given by Eq. (7.12). The values at $z = 0$, $r > 0$ are determined by the algebraic equation following from Eq. (7.7):

$$w_4(0, r) = M(0, r) \pm \{[M(0, r)]^2 + e^{2\beta(0,r)-2\alpha(0,r)}N(0, r)/r^2\}^{1/2}. \qquad (7.18)$$

At the $r = 0$ the differential equation (7.2c) reduces to the algebraic equation for w_4:

$$\begin{aligned} w_4(z, 0) = {} & (1/4)\, e^{2\beta^*-2B^*}\{(\alpha_1{}^* - 2\beta_1{}^* - \chi_1{}^*)\, H^* \\ & - [L^*L_1{}^* + (1/2)\, P_1{}^* - (4/3)\, e^{2B^*-2\chi^*}(\alpha^*_{12222} + 2(\alpha_1{}^* - \chi_1{}^*)\, \alpha^*_{2222})]/H^* \\ & + 4a^*e^{-2\chi^*}\chi_1{}^*(\beta_1{}^* + \chi_1{}^*)\} + M^* + 4\pi\rho^*a^*e^{-\chi^*+2\beta^*}, \end{aligned} \qquad (7.19)$$

where

$$\begin{aligned} H^* = {} & (L^{*2} + P^* - (8/3)\, e^{2B^*-2\chi^*}\alpha^*_{2222})^{1/2} \\ & \times \operatorname{sgn}\{4\pi\rho^*(0)\, a^*e^{-\chi^*(0)+2\beta^*(0)} + M^*(0) - w_4(0, 0)\}. \end{aligned} \qquad (7.20)$$

The sign of H^* is given by the same requirement that yields Eq. (7.14). The function $v_4(z, 0)$ is now given by Eq. (7.7) which takes here the simple form

$$v_4(z, 0) = L^* + H^*. \qquad (7.21)$$

By the substitution of the corresponding expressions for v_4 and v_{41} from Eq. (7.7), Eq. (7.2c) reduces to the quasilinear partial differential equation of the first order determining the function w_4 at $z > 0$, $r > 0$:

$$\{r(M - w_4)/H\}\, w_{41} + w_{42} = \Phi, \qquad (7.22)$$

where

$$\begin{aligned} \Phi = {} & (1/r)\big(w_4\{r(\beta_2 - \alpha_2 - \sigma_2 - \chi_2) - 4 - (2b_{00}/k^{44})[1 - 2a_0 + r(\sigma_2 + \chi_2)]\} \\ & + e^{2\beta-2\alpha}\{(L + H)[\alpha_1 - \beta_1 - (\sigma_1 + \chi_1)(1 + 2b_{00}/k^{44})] \\ & - L_1 + (2ae^{-2\chi}/k^{44})\{e^{2\beta-2\alpha}[\sigma_{11} + \chi_{11} - (\sigma_1 + \chi_1)(\alpha_1 - \beta_1 - \sigma_1 - \chi_1)] \\ & + \sigma_{22} + \chi_{22} - a_{00}(1 + b_{00}) + (\sigma_2 + \chi_2)(\alpha_2 - \beta_2 + \sigma_2 - \chi_2) \\ & + (\alpha_2 - \beta_2 + 2\sigma_2)/r - 4a_0{}^2\gamma + a_0(1 + b_{00})(\beta_2 - \alpha_2 + \sigma_2 + 3\chi_2)/r\} \\ & + 16\pi\rho e^{-\chi+2\beta}a/k^{44}) + (1/H)\big(w_4\{rw_4(\alpha_1 - \beta_1) \\ & - 2rM[\alpha_1 - \beta_1 - \chi_1 - (b_{00}/k^{44})(\sigma_1 + \chi_1)] \\ & - (2ae^{-2\chi}/k^{44})[\sigma_{12} + \chi_{12} + 2a_0b_{00}(\sigma_1 + \chi_1)/r]\} - (e^{2\beta-2\alpha}/r)(LL_1 + \tfrac{1}{2}N_1)\big). \end{aligned} \qquad (7.23)$$

and

$$H = [e^{2\alpha-2\beta}r^2w_4(2M - w_4) + L^2 + N]^{1/2} \operatorname{sgn} H^*(0). \qquad (7.24)$$

After the differential equation (7.22) has been multiplied by H, it becomes obvious that it is satisfied at $z = 0$, because $w_4(z, r)$ is an even function in z and $H(0, r) = 0$ (cf. Eqs. (7.12)–(7.14) and (7.24), (7.18), and the definition of L in Appendix C).

The integration of the quasilinear partial differential equation (7.22) can be reduced in the usual way [19] to the integration of the following two simultaneous ordinary differential equations

$$dw_4/dr = \Phi, \tag{7.25}$$

$$dz/dr = r(M - w_4)/H, \tag{7.26}$$

in which the independent variable is the coordinate r. The function w_4 and the coordinate z are dependent variables to be computed simultaneously by the numerical integration. The initial values at $r = 0$, given by Eq. (7.19), are

$$w_4 = w_4(z_i, 0), \qquad z = z_i, \tag{7.27}$$

with $z_i > 0$ indicating the coordinate z where the integration starts and

$$\Phi = 0, \qquad \text{at} \qquad r = 0, \tag{7.28}$$

because $w_4(z, r)$ is an even function in r and

$$dw_4/dr = w_{42} + w_{41}(dz/dr) = 0, \qquad \text{at} \qquad r = 0.$$

After the integration has been carried out, the values of $w_4(z_i, r)$ are known at $z = z(z_i, r)$. From these values the function w_4 at the prescribed points of each column of the two-dimensional grid used in the time evolution program (Sec. III) is to be computed by numerical interpolation taking into account that the function $w_4(z, r)$ is an even function in z.

Instead of integrating Eqs. (7.25) and (7.26) and successively eliminating the parameter z_i it is possible to proceed in the following way. The integration domain be covered by a two-dimensional grid of equally spaced points (the interval between two neighbouring points is to be chosen here smaller than the interval of the grid in the time evolution problem). At the first column of the grid points ($r = 0$) the values of w_4 are known [they are given by Eqs. (7.12) and (7.19)]. The values of w_{41} at each point of this column can be thus determined by numerical differentiation

fourth-order Runge-Kutta method for the integration of ordinary differential equations, we can now compute the values of w_4 at each point with $z > 0$ in the second column. At $z = 0$ the values $w_4(0, r)$ are given by Eq. (7.18). Henceforth the same procedure can be repeated. In this way the numerical integration of one partial differential equation (7.22) is reduced to the numerical integration of a system of N simultaneous ordinary differential equations, N being the number of points in one column.

After the function $w_4(z, r)$ has been determined at all the points of the two-dimensional grid of Sec. III, the function $v_4(z, r)$ is to be computed at these points with $z > 0$, $r > 0$ by the equation

$$v_4 = L + H, \tag{7.29}$$

following from Eq. (7.7). The function L is defined in Appendix C and the function H by Eq. (7.24).

Concluding, let us remark that the fourth, fifth, and sixth derivatives occurring in the equations of this section are the result of the limiting processes $r \to 0$, $z \to 0$.

Appendix A

The Determinant and the Contravariant Components of the Metric Tensor

Appendix B

The Functions $P_{\mu\nu}$ and P_{ϱ}

Appendix C

Definitions of Functions Occuring in Sec. VII

These three Appendices are deposited at ASIS/NAPS.[9]

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[9] See NAPS document # 02401 for 36 pages of the Appendices A, B, and C to this paper. Order from ASIS/NAPS, c/o Microfilm Publications, 305 East 46th Street, New York, N.Y. 10017. Remit with order for each NAPS document number \$1.50 for microfiche or \$5.00 for photocopies up to 30 pages, and 15¢ per pages for each additional pages over the first 30 pages. Make cheques payable to Microfiche Publications.

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